Electrification of an Insulated Lens, treated by the Stream-force-function; and Allied Problems.

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The chief object of this memoir was the calculation of the Stokes' stream-current or force-function, S, representing uniaxial surfaces having meridian section of a line of force, orthogonal to the uniaxial potential function, V, with special reference to the electrical problems discussed by Prof. H. M. Macdonald in the 'Proceedings of the London Mathematical Society' (L.M.S.), vol. 26, 1895, in a body bounded by spherical surfaces, as a lens.

1. The expression for V, the potential function (P.F.) of a lens, insulated and electrified to potential π, has been given by Macdonald in the form

$$V_0-V = \pi-V = C-D, \tag{1}$$

where, changing his η and ξ into u and v,

$$C = \int_u^\infty \frac{\sqrt{(\operatorname{ch} u-\cos v)}}{\sqrt{(\operatorname{ch}\zeta-\operatorname{ch} u)}}\,\frac{\operatorname{sh} f\zeta\,.\,fd\zeta}{\operatorname{ch} f\zeta-\cos fv}, \tag{2}$$

$$D = \int \frac{\sqrt{(\operatorname{ch} u-\cos v)}}{\sqrt{(\operatorname{ch}\zeta-\operatorname{ch} u)}}\,\frac{\operatorname{sh} f\zeta\,.\,fd\zeta}{\operatorname{ch} f\zeta-\cos f(v+2\beta)}, \tag{3}$$

so that C is derived from D by putting $\beta = 0$, or else D is derived from C by replacing fv by $f(v+2\beta)$.

Here, in the figure, u and v are the stereographic dipolar co-ordinates, connected with the Cartesian x and y by the conformal relation

$$y+ix = a\coth\tfrac{1}{2}(u+iv), \quad \text{or} \quad x+iy = a\cot\tfrac{1}{2}(v+iu), \tag{4}$$

$$x,\, y = a\,\frac{\sin v,\ \operatorname{sh} u}{\operatorname{ch} u-\cos v}, \tag{5}$$

and u is constant along a parallel of latitude on the stereographic projection of the hemisphere, with A and B as South and North Pole, while v is constant along a meridian, as on the figure.

In C and D of (2) and (3), $v = \alpha-\beta$ and $v = -\beta$ over the two boundary spherical surfaces of the lens, and $\alpha = \pi/f$ is the salient angle of the dielectric space outside the lens, the dielectric angle.

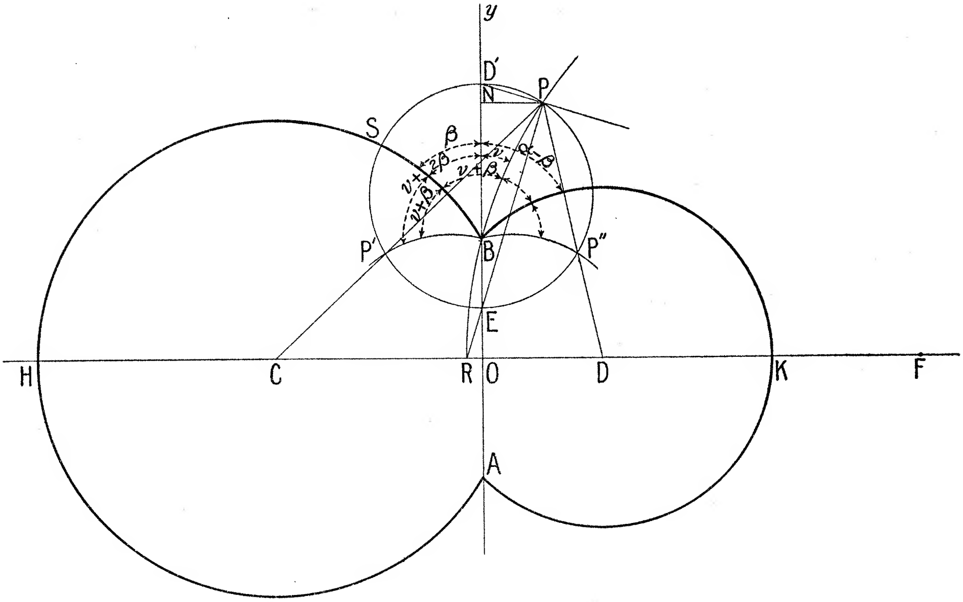
With C on the figure the centre of the circle $v = -\beta$,

$$CA = a\operatorname{cosec}\beta, \quad CP^2 = (x+a\cot\beta)^2+y^2 = \frac{\operatorname{ch} u-\cos(u+2\beta)}{\operatorname{ch} u-\cos v}\,CA^2, \tag{6}$$

$$D = \frac{CA}{CP} \cdot E, \text{ where } E = \int \sqrt{\left[\frac{\operatorname{ch} u - \cos(v+2\beta)}{\operatorname{ch}\zeta - \operatorname{ch} u}\right]} \frac{\operatorname{sh} f\zeta \cdot f d\zeta}{\operatorname{ch} f\zeta - \cos f(v+2\beta)}, \tag{7}$$

so that E is derived from C in (2) by a change of v throughout into $v+2\beta$.

Then since $f\alpha = \pi$, $\cos f(v+2\beta) = \cos fv$, $C = D$, $V = \pi$, over both the spherical surfaces of the lens, $v = \alpha-\beta$ and $v = -\beta$.



The circle ABP through P is defined by v, and thus $-v-2\beta$ defines the circle ABP′ through the inverse point, P′, in the sphere $v = -\beta$; and at the pole B, the circular arcs BP, BP′ make equal angles with the spherical surface $v = -\beta$.

2. We have to verify that V satisfies Laplace's equation. In these orthogonal conjugate function co-ordinates, u and v, Laplace's equation becomes

$$\frac{d}{du}\left(y\frac{dV}{du}\right) + \frac{d}{dv}\left(y\frac{dV}{dv}\right) = 0,$$

or

$$\frac{d}{du}\left(\frac{\operatorname{sh} u}{\operatorname{ch} u - \cos v} \cdot \frac{dV}{du}\right) + \frac{d}{dv}\left(\frac{\operatorname{sh} u}{\operatorname{ch} u - \cos v} \cdot \frac{dV}{dv}\right) = 0. \tag{A}$$

Here the differentiation of V, C, D in § 1 with respect to v does not introduce any difficulty. But with respect to u, there is the extra term to be considered due to the lower limit u, and this term is infinite, and will require to be cancelled by some other infinity under the integration.

We have then to write

$$\mathrm{C} = \int_u^\infty \mathrm{C}_1\, d\zeta, \qquad \frac{d\mathrm{C}}{du} = \int_u^\infty \left(\frac{d\mathrm{C}_1}{du} + \frac{d\mathrm{C}_1}{d\zeta}\right) d\zeta, \tag{1}$$

and trust to the cancelling of the infinities between $\dfrac{d\mathrm{C}_1}{d\zeta}$ and $\dfrac{d\mathrm{C}_1}{du}$.

In Hicks's manner for his toroidal function, putting

$$\mathrm{C} = \mathrm{P}\sqrt{(\operatorname{ch} u - \cos v)}, \qquad \mathrm{P} = \int \frac{d\zeta}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} \cdot \frac{f \operatorname{sh} f\zeta}{\operatorname{ch} f\zeta - \cos fv}, \tag{2}$$

Laplace's equation (A) then changes into

$$\frac{d^2\mathrm{P}}{du^2} + \coth u \frac{d\mathrm{P}}{du} + \frac{d^2\mathrm{P}}{dv^2} + \tfrac{1}{4}\mathrm{P} = 0, \quad \text{or} \quad \frac{d}{d\mu}(\mu^2 - 1)\frac{d\mathrm{P}}{d\mu} + \frac{d^2\mathrm{P}}{dv^2} + \tfrac{1}{4}\mathrm{P} = 0, \tag{3}$$

with $\operatorname{ch} u = \mu$, and the differentiations are simplified thereby.

In another conformal representation similar to (4), § 1,

$$z + iw = f \coth \tfrac{1}{2} f(\zeta + iv), \qquad z = \frac{f \operatorname{sh} f\zeta}{\operatorname{ch} f\zeta - \cos fv}, \tag{4}$$

$$\mathrm{C}_1 = \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} z, \qquad \mathrm{P}_1 = \frac{z}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}}, \qquad \mathrm{P} = \int_u^\infty \mathrm{P}_1\, d\zeta, \tag{5}$$

and we have to cancel the infinity arising at the lower limit u, making use of the differentiations

$$\begin{aligned} \frac{d\mathrm{P}}{du} &= \int \left(d\frac{\mathrm{P}_1}{d\zeta} + \frac{d\mathrm{P}_1}{du}\right) d\zeta \\ &= \int \frac{d\zeta}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} \left[\frac{dz}{d\zeta} - \tfrac{1}{2} z \coth \tfrac{1}{2}(\zeta + u)\right], \end{aligned} \tag{6}$$

$$\frac{d^2\mathrm{P}}{du^2} = \int \frac{d\zeta}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} \left[\frac{d^2z}{d\zeta^2} - \frac{dz}{d\zeta}\coth \tfrac{1}{2}(\zeta + u) + \tfrac{1}{2} z \operatorname{cosect}^2 \tfrac{1}{2}(\zeta + u) + \tfrac{1}{4} z\right], \tag{7}$$

and then

$$\frac{d^2\mathrm{P}}{du^2} + \coth u \frac{d\mathrm{P}}{du} + \tfrac{1}{4}\mathrm{P} = \int \frac{\mathrm{N}\, d\zeta}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}}, \tag{8}$$

$$\mathrm{N} = \frac{d^2z}{d\zeta^2} + \frac{dz}{d\zeta} \frac{\operatorname{sh}\tfrac{1}{2}(\zeta - u)}{\operatorname{sh} u \operatorname{sh}\tfrac{1}{2}(\zeta + u)} + \tfrac{1}{4} z \frac{-\operatorname{sh}\zeta + 2 \operatorname{sh} u}{\operatorname{sh} u \operatorname{sh}^2 \tfrac{1}{2}(\zeta + u)}, \tag{9}$$

$$\frac{\mathrm{N}}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} = \frac{1}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} \cdot \frac{d^2z}{d\zeta^2} + \frac{d}{d\zeta} \frac{\tfrac{1}{2} z [\operatorname{sh}\tfrac{1}{2}(\zeta - u)]^{1/2}}{\operatorname{sh} u [\operatorname{sh}\tfrac{1}{2}(\zeta + u)]^{3/2}}, \tag{10}$$

$$\frac{d^2\mathrm{P}}{du^2} + \coth u \frac{d\mathrm{P}}{du} + \tfrac{1}{4}\mathrm{P} = \int \frac{d\zeta}{\sqrt{(\operatorname{ch}\zeta - \operatorname{ch} u)}} \cdot \frac{d^2z}{d\zeta^2} + * \tag{11}$$

the star * denoting an integral which vanishes at both limits of ζ, u and ∞.

These precautions are not required in the differentiation with respect to v, so that Laplace's equation becomes

$$\int \frac{d\zeta}{\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}}\left(\frac{d^2z}{d\zeta^2}+\frac{d^2z}{dv^2}\right) = 0, \tag{12}$$

and so it is satisfied by P, and also by C, and the same procedure will serve for D.

3. Our chief object in this memoir is to discover S, the orthogonal stream-force function (S.F. or F.F.) of the P.F.V; a surface S = a constant containing the lines of force in a meridian section.

Laplace's equation in (A), §2, is equivalent to

$$\frac{dS}{du} = +y\frac{dV}{dv}, \qquad \frac{dS}{dv} = -y\frac{dV}{du}, \tag{1}$$

omitting Maxwell's factor 2π; and then

$$\frac{dV}{dv} = \frac{1}{y}\frac{dS}{du}, \qquad \frac{dV}{du} = -\frac{1}{y}\frac{dS}{dv}, \tag{2}$$

so that Laplace's equivalent equation for the S.F. is

$$\frac{d}{du}\left(\frac{1}{y}\frac{dS}{du}\right)+\frac{d}{dv}\left(\frac{1}{y}\frac{dS}{dv}\right) = 0. \tag{B}$$

With the stereographic co-ordinates of § 1, we have to put

$$S = \frac{R}{\sqrt{(\operatorname{ch}u-\cos v)}}, \tag{3}$$

to obtain the equation (B) in the form

$$\frac{d^2R}{du^2}-\coth u\frac{dR}{du}+\frac{d^2R}{dv^2}+\tfrac{1}{4}R = 0, \quad \text{or} \quad (\mu^2-1)\frac{d^2R}{d\mu^2}+\frac{d^2R}{dv^2}+\tfrac{1}{4}R = 0. \tag{4}$$

We may take for correspondence

$$S = \int z\sqrt{\left(\frac{\operatorname{ch}\zeta-\operatorname{ch}u}{\operatorname{ch}u-\cos v}\right)}\,d\zeta, \qquad R = \int z\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}\,d\zeta, \tag{5}$$

$$\frac{dR}{du} = \int\left[\frac{dz}{d\zeta}+\tfrac{1}{2}z\coth\tfrac{1}{2}(\zeta+u)\right]\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}\,d\zeta, \tag{6}$$

$$\frac{d^2R}{du^2} = \int\left[\frac{d^2z}{d\zeta^2}+\frac{dz}{d\zeta}\coth\tfrac{1}{2}(\zeta+u)-\tfrac{1}{2}z\operatorname{cosect}\tfrac{1}{2}(\zeta+u)+\tfrac{1}{4}z\right]\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}\,d\zeta, \tag{7}$$

$$\frac{d^2R}{du^2}-\coth u\frac{dR}{du}+\tfrac{1}{4}R = \int\left\{\frac{d^2z}{d\zeta^2}\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}+\frac{d}{d\zeta}\frac{[\tfrac{1}{2}z\operatorname{sh}\tfrac{1}{2}(\zeta-u)]^{3/2}}{\operatorname{sh}u\,[\operatorname{sh}\tfrac{1}{2}(\zeta+u)]^{1/2}}\right\}d\zeta$$

$$= \int\frac{d^2z}{d\zeta^2}\sqrt{(\operatorname{ch}\zeta-\operatorname{ch}u)}\,d\zeta+* \tag{8}$$

so that (4) is satisfied provided z is any conjugate function of ζ and v in (4), § 2.

4. Corresponding to the separate terms in V, C, and D, in § 1, we take

$$S = a(A-B), \tag{1}$$

satisfying the separate conditions

$$\frac{dA}{du} = -\frac{y}{a}\frac{dC}{dv}, \qquad \frac{dA}{dv} = +\frac{y}{a}\frac{dC}{du}, \tag{2}$$

$$\frac{dB}{du} = -\frac{y}{a}\frac{dD}{dv}, \qquad \frac{dB}{dv} = +\frac{y}{a}\frac{dD}{du}, \tag{3}$$

and then

$$\frac{dS}{adu} = \frac{dA}{du} - \frac{dB}{du} = -\frac{y}{a}\frac{dC}{dv} + \frac{y}{a}\frac{dD}{dv} = +\frac{y}{a}\frac{dV}{dv}, \tag{4}$$

$$\frac{dS}{adv} = -\frac{y}{a}\frac{dV}{du}. \tag{5}$$

There does not appear to be any direct method of writing down the integrals required for A, B, S. But by various conjectural considerations we are led to infer that we may take A, the S.F. of the P.F.C, of the form

$$A = \int_u^\infty A_1 f d\zeta, \tag{6}$$

$$A_1 = \sqrt{\left(\frac{\operatorname{ch}\zeta - \operatorname{ch} u}{\operatorname{ch} u - \cos v}\right)} \frac{\sin v}{\operatorname{ch}\zeta - \cos v} \cdot \frac{\operatorname{sh} f\zeta}{\operatorname{ch} f\zeta - \cos fv}$$
$$- \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} \frac{\operatorname{sh}\zeta}{\operatorname{ch}\zeta - \cos v} \cdot \frac{\sin fv}{\operatorname{ch} f\zeta - \cos fv}, \tag{7}$$

and derive B, the S.F. of D, by writing $f(v+2\beta)$ for fv;

$$B_1 = \sqrt{\left(\frac{\operatorname{ch}\zeta - \operatorname{ch} u}{\operatorname{ch} u - \cos v}\right)} \frac{\sin v}{\operatorname{ch}\zeta - \cos v} \cdot \frac{\operatorname{sh} f\zeta}{\operatorname{ch} f\zeta - \cos f(v+2\beta)}$$
$$- \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} \frac{\operatorname{sh}\zeta}{\operatorname{ch}\zeta - \cos v} \cdot \frac{\sin f(v+2\beta)}{\operatorname{ch} f\zeta - \cos f(v+2\beta)}. \tag{8}$$

We have to verify then that

$$\frac{a}{y}\frac{dA}{du} + \frac{dC}{dv} = 0, \qquad \frac{a}{y}\frac{dA}{dv} - \frac{dC}{du} = 0, \qquad \frac{y}{a} = \frac{\operatorname{sh} u}{\operatorname{ch} u - \cos v}. \tag{9}$$

Performing the differentiation, we find

$$\frac{\operatorname{ch} u - \cos v}{\operatorname{sh} u}\frac{dA_1}{du} + \frac{dC_1}{dv} = \frac{d}{d\zeta}\sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} \frac{\sin fv}{\operatorname{ch} f\zeta - \cos fv}, \tag{10}$$

and here an infinity arises at the lower limit u, to be cancelled by writing

$$\frac{\operatorname{ch} u - \cos v}{\operatorname{sh} u}\left(\frac{dA_1}{d\zeta} + \frac{dA_1}{du}\right) + \frac{dC_1}{dv}$$
$$= \frac{d}{d\zeta}\frac{\sqrt{(\operatorname{ch} u - \cos v \,.\, \operatorname{ch}\zeta - \operatorname{ch} u)}}{\operatorname{sh} u} \cdot \frac{\sin v}{\operatorname{ch}\zeta - \cos v} \cdot \frac{\operatorname{sh} f\zeta}{\operatorname{ch} f\zeta - \cos fv}$$
$$+ \frac{d}{d\zeta}\sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)}\left(1 - \frac{\operatorname{sh}\zeta}{\operatorname{sh} u} \cdot \frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \cos v}\right) \frac{\sin fv}{\operatorname{ch} f\zeta - \cos fv}, \tag{11}$$

in which the second line reduces to

$$\frac{d}{d\zeta}\frac{\sqrt{(\operatorname{ch} u-\cos v\,.\,\operatorname{ch}\zeta-\operatorname{ch} u)}}{\operatorname{sh} u}\cdot\frac{\operatorname{ch}\tfrac{1}{2}(\zeta-u)-\operatorname{ch}\tfrac{1}{2}(\zeta+u)\cos v}{\operatorname{sh}\tfrac{1}{2}(\zeta+u)(\operatorname{ch}\zeta-\cos v)}\cdot\frac{\sin fv}{\operatorname{ch} f\zeta-\cos fv}, \tag{12}$$

so that, integrating with respect to ζ,

$$\frac{\operatorname{ch} u-\cos v}{\operatorname{sh} u}\cdot\frac{dA}{du}+\frac{dC}{du}=*+*=0, \tag{13}$$

provided $f>\frac{1}{2}$, $\alpha<2\pi$ as required in a physical problem.

Proceeding in the same way with the other differentiations, we find

$$\frac{\operatorname{ch} u-\cos v}{\operatorname{sh} u}\cdot\frac{dA_1}{dv}-\frac{dC_1}{d\zeta}-\frac{dC_1}{du}$$

$$=\frac{d}{d\zeta}\sqrt{\left(\frac{\operatorname{ch} u-\cos v}{\operatorname{ch}\zeta-\operatorname{ch} u}\right)}\left(\frac{\operatorname{sh}\zeta}{\operatorname{sh} u}\cdot\frac{\operatorname{ch} u-\cos v}{\operatorname{ch}\zeta-\cos v}-1\right)\frac{\operatorname{sh} f\zeta}{\operatorname{ch} f\zeta-\cos fv}$$

$$=\frac{d}{d\zeta}\frac{\sqrt{(\operatorname{ch} u-\cos v\,.\,\operatorname{ch}\zeta-\operatorname{ch} u)}}{\operatorname{sh} u}\cdot\frac{\operatorname{ch}\tfrac{1}{2}(\zeta+u)\cos v-\operatorname{ch}\tfrac{1}{2}(\zeta-u)}{\operatorname{sh}\tfrac{1}{2}(\zeta+u)(\operatorname{ch}\zeta-\cos v)}\cdot\frac{\operatorname{sh} f\zeta}{\operatorname{ch} f\zeta-\cos fv}, \tag{14}$$

and integrating with respect to ζ,

$$\frac{\operatorname{ch} u-\cos v}{\operatorname{sh} u}\frac{dA}{dv}-\frac{dC}{du}=*=0, \tag{15}$$

so that the conditions in (9) are both satisfied.

Similarly for B and D.

5. To verify our conjectures concerning A and B, we test the preceding formulas on a few of the simplest cases.

Begin with the complete sphere, for which $f=1$, and the dielectric angle is two right angles, $\alpha=180°$. Then in (2) § 1,

$$C=\pi, \qquad E=\pi, \qquad V=\pi\frac{CA}{CP}. \tag{1}$$

In the S.F., $A=0$; and with $\operatorname{ch}\zeta=s$, $\operatorname{ch} u=b$,

$$B=\int_b^\infty\sqrt{\left(\frac{s-b}{b-\cos v}\right)}\frac{\sin v\,ds}{s-\cos v\,.\,s-\cos(v+2\beta)}$$

$$-\int\sqrt{\left(\frac{b-\cos v}{s-b}\right)}\frac{\sin(v+2\beta)\,ds}{s-\cos v\,.\,s-\cos(v+2\beta)}$$

$$=-\int\sqrt{\left(\frac{b-\cos v}{s-b}\right)}\frac{\cot\beta\,ds}{s-\cos v}$$

$$+\frac{\operatorname{ch} u\cos\beta-\cos(v+\beta)}{\sin\beta\sqrt{(b-\cos v)}}\int\frac{ds}{\sqrt{(s-b)}}\cdot\frac{1}{s-\cos(v+2\beta)}. \tag{2}$$

Making use of the lemmas of the Integral Calculus

$$\int_b^\infty \frac{ds}{\sqrt{(s-b)}} \cdot \frac{1}{s-c} = \frac{\pi}{\sqrt{(b-c)}}, \quad \text{and} \quad \int \frac{ds}{\sqrt{(s-b)}} \cdot \frac{1}{(s-c)^2} = \frac{\frac{1}{2}\pi}{(b-c)^{3/2}}, \tag{3}$$

by a differentiation with respect to c,

$$\mathrm{B} = -\pi \cot\beta + \frac{\pi}{\sin\beta} \cdot \frac{\operatorname{ch} u \cos\beta - \cos(v+\beta)}{\sqrt{[\operatorname{ch} u - \cos v \,.\, \operatorname{ch} u - \cos(v+2\beta)]}}$$

$$= -\pi\cot\beta + \frac{\mathrm{CA}}{\mathrm{OA}}\cos \mathrm{OCP}, \tag{4}$$

$$\mathrm{S} = a(\mathrm{A}-\mathrm{B}) = \pi \,.\, \mathrm{OA} - \pi \,.\, \mathrm{CA}\cos\mathrm{OCP}, \tag{5}$$

of which the constant term may be ignored.

6. Next for two orthogonal spherical surfaces, as shown in the figure, with centres at C and D; $f = 2$, and the dielectric angle is a right angle, $\alpha = 90°$.

With $\operatorname{ch}\zeta = s$, here

$$\mathrm{D} = \int \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{s - \operatorname{ch} u}\right)} \frac{2s\,ds}{s^2 - \cos^2(v+2\beta)}$$

$$= \int \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{s - \operatorname{ch} u}\right)} \left[\frac{ds}{s - \cos(v+2\beta)} + \frac{ds}{s + \cos(v+2\beta)}\right]$$

$$= \pi\sqrt{\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u - \cos(v+2\beta)}} + \pi\sqrt{\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u + \cos(v+2\beta)}}$$

$$= \pi\frac{\mathrm{CA}}{\mathrm{CP}} + \pi\frac{\mathrm{DA}}{\mathrm{DP}}; \tag{1}$$

and putting $\beta = 0$,

$$\mathrm{C} = \pi + \pi\sqrt{\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u + \cos v}} = \pi + \pi\frac{\mathrm{OA}}{\mathrm{OP}}, \tag{2}$$

$$\frac{\mathrm{V}}{\pi} = 1 - \frac{\mathrm{C}-\mathrm{D}}{\pi} = +\frac{\mathrm{CA}}{\mathrm{CP}} + \frac{\mathrm{DA}}{\mathrm{DP}} - \frac{\mathrm{OA}}{\mathrm{OP}}, \tag{3}$$

the potential of the three electric images at C, D, O.

Over the surface AHB, centre C, and D, O inverse points in this sphere, $\frac{\mathrm{OA}}{\mathrm{OP}} = \frac{\mathrm{DA}}{\mathrm{DP}}$, while $\mathrm{CA} = \mathrm{CP}$, $\mathrm{V} = \pi$; and over AKB, centre D, and C, O inverse points, $\frac{\mathrm{OA}}{\mathrm{OP}} = \frac{\mathrm{CA}}{\mathrm{CP}}$, $\mathrm{DA} = \mathrm{DP}$, $\mathrm{V} = \pi$.

For the S.F., with $\operatorname{ch} u = b$, $\cos v = c$,

$$A = \int \sqrt{\left(\frac{s-b}{b-c}\right)} \frac{\sin v}{s-c} \cdot \frac{4s\,ds}{2(s^2-c^2)} - \int \sqrt{\frac{b-c}{s-b}} \frac{\sin 2v}{s-c} \cdot \frac{2\,ds}{2(s^2-c^2)}$$

$$= \frac{2\sin v}{\sqrt{(b-c)}} \int \frac{ds}{\sqrt{(s-b)}} \cdot \frac{s^2-bs-bc+c^2}{(s-c)^2(s+c)}$$

$$= \frac{2\sin v}{\sqrt{(b-c)}} \int \frac{ds}{\sqrt{(s-b)}} \left[-\frac{a-c}{(s-c)^2} + \frac{\frac{1}{2}}{s-c} + \frac{\frac{1}{2}}{s+c}\right]$$

$$= \frac{2\sin v}{\sqrt{(b-c)}} \left[\frac{-\frac{1}{2}\pi(a-c)}{(a-c)^{3/2}} + \frac{\frac{1}{2}\pi}{\sqrt{(a-c)}} + \frac{\frac{1}{2}\pi}{\sqrt{(a+c)}}\right]$$

$$= \frac{\pi \sin v}{\sqrt{(b^2-c^2)}} = \frac{\pi \sin v}{\operatorname{ch} u - \cos v} \cdot \sqrt{\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u + \cos v}} = \frac{\pi x}{\mathrm{OP}} = \pi \cos x\mathrm{OP}. \qquad (4)$$

And then for B, with $\cos(v+2\beta) = c'$,

$$B = \int \sqrt{\left(\frac{s-b}{b-c}\right)} \frac{\sin v}{s-c} \frac{4s\,ds}{2(s^2-c'^2)} - \int \sqrt{\left(\frac{b-c}{s-b}\right)} \frac{\sin(v+2\beta)}{s-c} \cdot \frac{4c'\,ds}{2(s^2-c'^2)}, \qquad (5)$$

$$aB = -2\pi a \cot 2\beta + \pi \,.\, \mathrm{CA} \,.\, \cos \mathrm{OCP} + \pi \,.\, \mathrm{DA} \,.\, \cos \mathrm{ODP}, \qquad (6)$$

by the reductions as above and ignoring the constant term $2\pi a \cot 2\beta$,

$$S = a(A-B) = \pi \,.\, \mathrm{OA} \cos x\mathrm{OP} - \pi \,.\, \mathrm{CA} \,.\, \cos \mathrm{OCP} - \pi \,.\, \mathrm{DA} \cos \mathrm{ODP}, \qquad (7)$$

and here the three terms represent the S.F. of the three images at O, C, D, with capacity represented by the radius OA, CA, DA.

Generally, for f an integer, as in L.M.S., vol. 26, p. 167, the P.F. and S.F. of the lens is due to a series of f images, arranged along the line of centres.

7. Next for a spherical bowl, AHB in the figure; with $f = \frac{1}{2}$, and dielectric angle, $\alpha = 360°$. Then, with $\operatorname{ch} \frac{1}{2}\zeta = s$,

$$C = \int_u^\infty \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} \zeta - \operatorname{ch} u}\right)} \frac{\operatorname{sh} \frac{1}{2}\zeta \,.\, \frac{1}{2} d\zeta}{\operatorname{ch} \frac{1}{2}\zeta - \cos \frac{1}{2} v} = \int \sqrt{\left(\frac{\operatorname{ch}^2 \frac{1}{2} u - \cos^2 \frac{1}{2} v}{s - \operatorname{ch}^2 \frac{1}{2} u}\right)} \frac{ds}{s - \cos \frac{1}{2} v}$$

$$= \sin^{-1} \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u + 1}\right)} \frac{\sqrt{(s^2 - \operatorname{ch}^2 \frac{1}{2} u)}}{s - \cos \frac{1}{2} v}\Bigg]_u^\infty$$

$$= \sin^{-1} \sqrt{\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u + 1}} = \sin^{-1} \frac{\mathrm{AB}}{\mathrm{AP} + \mathrm{PB}} = \omega, \qquad (1)$$

the P.F. of the circular disc on AB, when electrified to potential $\frac{1}{2}\pi$.

Similarly, writing $\operatorname{ch} \frac{1}{2} u = b$, $\cos \frac{1}{2}(v+2\beta) = c'$,

$$E = \int \sqrt{\left(\frac{b^2-c'^2}{s^2-b^2}\right)} \frac{ds}{s-c'} = \sin^{-1} \sqrt{\left(\frac{b^2-c'^2}{b^2}\right)} \frac{\sqrt{(s^2-b^2)}}{s-c}\Bigg]_b^\infty$$

$$= \sin^{-1} \sqrt{\left(1 - \frac{c'^2}{b^2}\right)} = \cos^{-1} \frac{c'}{b} = \omega', \qquad (2)$$

suppose, and $D = \dfrac{\mathrm{CA}}{\mathrm{CP}} \omega'$.

But if P′ is the point inverse to P in the spherical surface AHB, $v = -\beta$,

$$\frac{\mathrm{AB}}{\mathrm{AP'+P'B}} = \frac{\mathrm{AB}}{(\mathrm{CA/CP})\,.\,\mathrm{AP}+(\mathrm{CB/CP})\,\mathrm{PB}} = \frac{\mathrm{CP}}{\mathrm{CA}}\sin\omega$$

$$= \sqrt{\frac{\operatorname{ch} u - \cos(v+2\beta)\,.\,\operatorname{ch} u - \cos v}{\operatorname{ch} u - \cos v\,.\,\operatorname{ch} u + 1}} = \sqrt{\frac{\operatorname{ch} u - \cos(v+2\beta)}{\operatorname{ch} u + 1}}$$

$$= \sqrt{\frac{b^2 - c'^2}{b^2}} = \sin\omega', \quad (3)$$

and ω' is the P.F. at P′ of the electrified disc on AB.

When the bowl is electrified to potential π, this makes

$$\mathrm{V} = \pi - \omega + \frac{\mathrm{CA}}{\mathrm{CP}}\,\omega', \quad (4)$$

instead of the usual form given, due to taking a different starting point for the multiple values of ω and ω'.

Draw the co-axial circle through P, P′, crossing the sphere of the bowl at S, S′; and let P, P′ circulate round this circle in opposite directions.

Starting with P and P′ in coincidence at S on the bowl, where CP = CP′ = CA, the multiplicity is chosen so as to make $\omega = \omega'$ at S, $\mathrm{V} = \pi$. Then P, P′ proceed in opposite directions, and cross again at S′, where CP = CP′ = CA, but $\omega' = \pi - \omega$; thus over the blank zone AKB of the spherical surface, $\mathrm{V} = 2\omega'$, double of the P.F. of the electrified disc AB, and this is not constant.

This verifies when the bowl is a complete sphere, and AB = 0. For if P is an internal point, $\omega = \omega' = 0$, $\mathrm{V} = \pi$; but, if P is external, $\omega = \omega' = \pi$, $\mathrm{V} = \pi\dfrac{\mathrm{CA}}{\mathrm{CP}}$.

Proceeding with the S.F. for $f = \frac{1}{2}$, and first for A, from (7), § 4, with $\cos\frac{1}{2}v = e$,

$$\mathrm{A} = \int\sqrt{\left(\frac{s^2-b^2}{b^2-e^2}\right)}\cdot\frac{\sin v\,ds}{2(s^2-e^2)(s-e)} - \int\sqrt{\left(\frac{b^2-e^2}{s^2-b^2}\right)}\frac{\sin\frac{1}{2}v\,.\,2s\,ds}{2(s^2-e^2)(s-e)}$$

$$= \frac{\sin\frac{1}{2}v}{\sqrt{(b^2-e^2)}}\int\frac{ds}{\sqrt{(s^2-b^2)}}\cdot\frac{es-b^2}{(s-e)^2}$$

$$= \frac{\sin\frac{1}{2}v}{\sqrt{(b^2-e^2)}}\left[-\frac{\sqrt{(s^2-b^2)}}{s-e}\right]_b^\infty = -\frac{\sin\frac{1}{2}v}{\sqrt{(b^2-e^2)}} = -\sqrt{\frac{1-\cos v}{\operatorname{ch} u - \cos v}}$$

$$= -\sqrt{\left(1-\frac{\operatorname{ch} u - 1}{\operatorname{ch} u - \cos v}\right)} = -\sqrt{\left[1-\left(\frac{r_1-r_2}{2a}\right)^2\right]}, \quad (6)$$

so that $-\mathrm{A}a$ is the semi-conjugate axis of the hyperbola through P, foci at A, B.

Next, from (8), § 4, for B, with $\cos \frac{1}{2}(v+2\beta) = c$,

$$\mathrm{B} = \int \sqrt{\left(\frac{s^2-b^2}{b^2-e^2}\right)} \frac{\sin v \, ds}{2(s^2-e^2)(s-c)} - \int \sqrt{\left(\frac{b^2-e^2}{s^2-b^2}\right)} \frac{\sin \frac{1}{2}(v+2\beta) \, . \, 2s \, ds}{2(s^2-e^2)(s-b)}$$

$$= \frac{1}{\sqrt{(b^2-c^2)}} \int \frac{ds}{\sqrt{(s^2-a^2)}} \left(\frac{\mathrm{L}_1}{s-e} + \frac{\mathrm{L}_2}{s+e} + \frac{\mathrm{L}_3}{s-b}\right), \tag{7}$$

and, working out the partial fractions and integration, we find, after reduction,

$$\mathrm{B}a = \mathrm{CA}(\omega' \cos \mathrm{PCO} - \omega), \tag{8}$$

the S.F. of D, agreeing with the result obtained by J. R. Wilton in the ‘Messenger of Mathematics,’ August, 1914.

8. In the geometry of the electrical inversion of the bowl with respect to a point, Q, on the rim, to obtain the electrification induced in a semi-infinite plate by a point charge at Q, I have to thank the kind assistance of Mr. C. N. H. Lock, Fellow of Caius College.

The plane through a point, P, and the axis OC of the bowl, is shown in the figure, where the bisector, PR, of the angle APB crosses AB in E, and the axis OC in R, on the circle round APB, centre F.

The triangles EAP, BRP are similar, because EAP = BRP, EPA = BPR; and then

$$\sin \omega = \frac{\mathrm{AB}}{\mathrm{AP}+\mathrm{PB}} = \frac{\mathrm{AE}}{\mathrm{AP}} = \frac{\mathrm{EB}}{\mathrm{PB}} = \frac{\mathrm{BR}}{\mathrm{PR}}. \tag{1}$$

Next invert with respect to any point, Q, on the circular edge, ABQ, of the bowl, with $\mathrm{QO}_1 = \mathrm{AB}$, the diameter the base, as the radius of inversion.

The circle QAB inverts into the edge $\mathrm{O}_1 z$ of the semi-infinite plate, and in the plane of the figure through Q and the axis OC of the bowl, $\mathrm{O}_1 x$ perpendicular to QC is the trace of the plate $x\mathrm{O}_1 z$, inverse of the bowl.

The circle, centre O_1 and radius $\mathrm{O}_1\mathrm{Q}$, is the inverse of the axis OC; and if R_1 is the inverse point of R, $\mathrm{O}_1\mathrm{R}_1$ is the trace of the plane $\mathrm{R}_1\mathrm{O}_1\mathrm{P}_1$, inverse of the sphere through P, centre F; and P_1 being the inverse point of P,

$$\frac{\mathrm{QP}_1}{\mathrm{P}_1\mathrm{R}_1} = \frac{\mathrm{QR}}{\mathrm{PR}} = \frac{\mathrm{BR}}{\mathrm{PR}} = \frac{\mathrm{AE}}{\mathrm{AP}} = \sin \omega. \tag{2}$$

This proves almost all that is required. Because, in electrical inversion, if ω is a P.F. at P, then

$$\mathrm{QP} \, . \, \omega = \frac{\mathrm{QO}_1{}^2}{\mathrm{QP}_1} \omega = \frac{\mathrm{AB}^2}{\mathrm{QP}_1} \sin^{-1} \frac{\mathrm{QP}_1}{\mathrm{P}_1\mathrm{R}_1}, \tag{3}$$

is a P.F. at P_1.

For the second part, $\dfrac{\mathrm{CA}}{\mathrm{CP}} \omega'$, in the P.F. of the bowl, draw the figure in the plane QPP′C, passing through QC, and so at right angles to the plate.

Then CP.CP′ = CQ², so that the circle round PQP′ touches CQ at Q, and with P_1, P_1', the inverse points of P, P′, inverted from Q, a circle goes round $PP'P_1'P_1$, and then $QP_1'P_1 = QPP' = P'QC$, so that P_1P_1' is parallel to CQ, and so at right angles to the plate.

With Q′, the inverse point of the centre C of the bowl, optical image of Q in the plate,

$$\frac{Q'P_1'}{CP'} = \frac{QQ'}{QP'}, \text{ while } \frac{CP'}{QP'} = \frac{QC}{QP}, \tag{4}$$

because P, P′ are inverse points in the bowl; thence

$$Q'P_1' = \frac{QQ'.CP'}{QP'} = \frac{QQ'.QC}{QP} = \frac{QO_1^2}{QP} = QP_1, \tag{5}$$

so that P_1P_1' is bisected at right angles by the plate; and P_1, P_1' are thus optical images in the plate, a general theorem.

Then with $\dfrac{CP'}{CA}\,\omega'$, a P.F. at P, a P.F. at the inverse point P_1 is

$$\frac{QP.CP'}{CQ}.\omega' = QP'.\omega' = \frac{AB^2}{QP_1'}\sin^{-1}\frac{QP_1'}{P_1'R_1'} = \frac{AB^2}{Q'P_1}\sin^{-1}\frac{Q'P_1}{R_1P_1}; \tag{6}$$

because $QP_1' = Q'P_1$, $P_1'R_1' = R_1P_1$.

And since PE.ER = DE.EO = P′E.ER′, a circle goes round PRR′P′; and CR.CR′ = CA², so that R_1, R_1' are optical images in the plate.

The circle RBP through P is defined by v, and then $-v-2\beta$ defines the circle R′BP′ through P′; thus, at the point B on the rim, the circular arcs BP, BP′ make equal angles with the bowl BS, the curvilinear angles at B, PBS and SBP′, being equal. And AK bisects RAR′, where R, R′ are inverse points in the bowl.

Then in the inversion, the planes O_1R_1z, $O_1R_1'z$ make equal angles with OC_1, trace of the plane inverse of the bowl; and since R_1, R_1' lie on the circle, centre O_1, and radius O_1Q_1, they are optical images in this plane.

We infer from the inversion that

$$V_1 = \frac{1}{P_1Q}\sin^{-1}\frac{P_1Q}{P_1R_1} - \frac{1}{P_1Q'}\sin^{-1}\frac{P_1Q'}{P_1R_1}, \tag{7}$$

is a potential function, zero over the semi-infinite plate, and so is the P.F. of the electrification induced by a point charge at Q.

To obtain this result from the form given by Macdonald in L.M.S., vol. 26, p. 163, replace on p. 162 his

$$2\tan^{-1}\sqrt{\frac{\operatorname{ch}\frac{1}{2}\eta-\cos\frac{1}{2}(\theta-\theta')}{\operatorname{ch}\frac{1}{2}\eta+\cos\frac{1}{2}(\theta-\theta')}} \quad\text{by}\quad \cos^{-1}\frac{\cos\frac{1}{2}(\theta-\theta')}{\operatorname{ch}\frac{1}{2}\eta}$$

or

$$\cos^{-1}\sqrt{\frac{1+\cos(\theta-\theta')}{\operatorname{ch}\eta+1}} = \sin^{-1}\sqrt{\frac{\operatorname{ch}\eta-\cos(\theta-\theta')}{\operatorname{ch}\eta+1}}, \tag{8}$$

and then, from p. 161,

$$2rr' \operatorname{ch} \eta = r^2+r'^2+z^2, \tag{9}$$

$$2rr' (\operatorname{ch} \eta+1) = (r+r')^2+z^2 = P_1R_1^2, \tag{10}$$

$$2rr' [\operatorname{ch} \eta-\cos(\theta-\theta')] = QP_1^2, \tag{11}$$

thus leading to the form above in (2), § 1, and to Sommerfeld's result.

The generalisation is obvious to Macdonald's result on p. 162, where $f = m+\frac{1}{2}$, half an odd integer; then for a point charge at Q in the dielectric space between two conducting planes to earth, at an angle $\dfrac{\pi}{f} = \dfrac{2\pi}{2m+1}$, the $4m+2$ optical images, Q_k, must be taken, including Q at Q_1; and the potential V_1 at P_1 of the induced electrification is given by

$$\frac{\pi V_1}{q} = \sum_{k=0}^{k=2m} \frac{(-1)^k}{P_1Q_k} \sin^{-1} \frac{P_1Q_k}{P_1R_1}. \tag{12}$$

Inversion with respect to Q will give a lens again with dielectric angle $\dfrac{2\pi}{2m+1}$; and when electrified to potential π, the potential V at an external point P in the dielectric space will be given in the form

$$V = \pi-\Sigma(-1)^k \frac{CA}{CP_k} \omega_k, \tag{13}$$

where ω_k is the P.F. at P_k of the disc on AB, electrified to potential $\frac{1}{2}\pi$, $\omega_k = \sin^{-1} \dfrac{AB}{AP_k+P_kB}$.

9. The expression for the capacity, K, of a lens employed by Macdonald can be utilised as a verification of our expressions for the S.F., S.; this should give the same value to K, by the difference of the values of S, S_1, and S_2 at H and K, vertex on the axis of the lens, in the form

$$K = \frac{S_1-S_2}{2\pi}. \tag{1}$$

At H, $u = 0$, $v = -\beta$, $v+2\beta = \beta$,

$$S_1 = 2a \sin f\beta \sqrt{(1-\cos\beta)} \int \frac{\sqrt{(\operatorname{ch} \zeta+1)} f d\zeta}{(\operatorname{ch} \zeta-\cos\beta)(\operatorname{ch} f\zeta-\cos f\beta)}, \tag{2}$$

and similarly at K, where $u = 0$, $v = \alpha-\beta$, $v+2\beta = \alpha+\beta$,

$$S_2 = 2a \sin f\beta \sqrt{[1-\cos(\alpha-\beta)]} \int \frac{\sqrt{(\operatorname{ch} \zeta+1)} f d\zeta}{[\operatorname{ch} \zeta-\cos(\alpha-\beta)](\operatorname{ch} f\zeta+\cos f\beta}. \tag{3}$$

Tested for the complete sphere, with $f = 1$, $\alpha = 180°$,

$$S_1 = 2a \sin\beta \sqrt{(1-\cos\beta)} \int \frac{\sqrt{(\operatorname{ch}\zeta+1)}\, d\zeta}{(\operatorname{ch}\zeta-\cos\beta)^2}$$

$$= 2a \sin\beta \sqrt{(1-\cos\beta)} \int_1^\infty \frac{ds}{\sqrt{(s-1)}\,(s-\cos\beta)^2} = \frac{\pi a \sin\beta}{1-\cos\beta}, \quad (4)$$

$$S_2 = \frac{-\pi a}{1+\cos\beta}, \quad S_1 - S_2 = \frac{2\pi a}{\sin\beta} = 2\pi \,.\, \mathrm{CA}, \quad (5)$$

and the capacity K is the radius, CA.

For two orthogonal surfaces, $f = 2$, $\alpha = 90°$,

$$S_1 = 2a \sin 2\beta \sqrt{(1-\cos\beta)} \int \frac{\sqrt{(\operatorname{ch}\zeta+1)}\, 2d\zeta}{(\operatorname{ch}\zeta-\cos\beta)(\operatorname{ch} 2\zeta - \cos 2\beta)}$$

$$= 4a \sin 2\beta \sin \tfrac{1}{2}\beta \int \frac{ds}{\sqrt{(s-1)}} \cdot \frac{1}{(s-\cos\beta)^2 (s+\cos\beta)}$$

$$= \pi a (\operatorname{cosec}\beta + \cot\beta - \tan\beta + \sec\beta - 1), \quad (6)$$

$$S_2 = -\pi a (\sec\beta + \tan\beta - \cot\beta + \operatorname{cosec}\beta - 1), \quad (7)$$

$$K = \frac{S_1 - S_2}{2\pi} = a (\operatorname{cosec}\beta + \sec\beta - 1) = \mathrm{CA} + \mathrm{DA} - \mathrm{OA}. \quad (8)$$

For the bowl, $f = \frac{1}{2}$, $\alpha = 360°$, and with $\operatorname{ch}\frac{1}{2}\zeta = s$, $\cos\frac{1}{2}v = c$,

$$S_1 = 2a(1-c^2) \int \frac{ds}{\sqrt{(s^2-1)}} \cdot \frac{s}{(s-c)^2 (s+c)}, \quad (9)$$

$$S_2 = -2a(1-c^2) \int \frac{ds}{\sqrt{(s^2-1)}} \cdot \frac{s}{(s-c)(s+c)^2}, \quad (10)$$

$$S_1 - S_2 = 4a(1-c^2) \int_1^\infty \frac{ds}{\sqrt{(s^2-1)}} \cdot \frac{s^2}{(s^2-c^2)^2}, \quad (11)$$

$$K = \frac{S_1 - S_2}{2\pi} = \frac{a}{2\pi} \cdot \frac{\pi - \beta + \sin(\pi-\beta)}{\sin\beta}$$

$$= \frac{\text{arc AHB} + \text{chord AOB}}{2\pi} = \frac{\text{girth}}{2\pi}, \quad (12)$$

as on the figure, and this verifies for the complete sphere, and the disc.

10. Encouraged by the verifications of our conjecture, for $f = 1, 2, \frac{1}{2}$, we proceed to test the S.F. method on the lens with conducting angle 90°, $\alpha = \frac{3}{2}\pi$, $f = \frac{2}{3}$; and for the hemisphere take $\beta = -\pi$, $\alpha - \beta = \frac{1}{2}\pi$.

As before, with $\operatorname{ch}\frac{1}{3}\zeta = s$, $\cos\frac{1}{3}\beta = \cos\gamma = b$, $1 - \cos\beta = (1-b)(1+2b)^2)$ $b_1 = \cos(\gamma + \frac{2}{3}\pi)$, $b_2 = \cos(\gamma + \frac{4}{3}\pi)$, $b_3 = \cos(\gamma+\pi) = -b$,

$$S_1 = \tfrac{1}{2} a \sqrt{(1-\cos 3\gamma)} \sin 2\gamma \int \frac{ds}{\sqrt{(s-1)}} \cdot \frac{2s-1}{(s-b)^2 (s-b_1)(s-b_2)(s-b_3)}, \quad (1)$$

and changing γ into $\frac{1}{2}\pi-\gamma$, with

$$c = \sin\gamma,\ c_1 = \sin(\gamma+\tfrac{2}{3}\pi),\ c_2 = \sin(\gamma+\tfrac{7}{3}\pi), c_3 = \sin(\gamma+\pi) = --c,$$

$$S_2 = -\tfrac{1}{2}a\sqrt{(1+\sin 3\gamma)}\sin 2\gamma\int\frac{ds}{\sqrt{(s-1)}}\cdot\frac{2s-1}{(s-c)^2(s-c_1)(s-c_2)(s-c_3)}. \qquad (2)$$

For S_1, the resolution is required into the partial fractions of

$$\frac{2s-1}{(s-b)^2(s-b_1)(s-b_2)(s-b_3)} = \frac{L_1}{(s-b)^2}+\frac{L_2}{s-b}+\frac{L_3}{s-b_1}+\frac{L_4}{s-b_2}+\frac{L_5}{s+b}, \qquad (3)$$

with $b = \cos\gamma$ replaced by $c = \sin\gamma$ for S_2.

The work was heavy and the details are omitted here, but the verification with Macdonald's result was effected.

But it is advisable to put on record the values of L, or rather of the corresponding integrals of the partial fractions, as it did not prove an easy matter to get them right; so making use of the lemma

$$1-\cos 3\gamma = (1+2b)^2(1-b) = (1+2b_1)^2(1-b_1) = \ldots, \qquad (4)$$

$$\frac{S_1}{\pi} = N_1+N_2+N_3+N_4+N_5, \qquad (5)$$

$$N_1 = \frac{1+\cos\gamma}{6\sin\gamma}, \qquad (6)$$

$$N_2 = -\frac{\sin\gamma}{6\cos\gamma}+\frac{1-\cos 2\gamma}{3\sin 3\gamma}, \qquad (7)$$

$$N_3+N_4 = \frac{2\sin 2\gamma}{\sqrt{3}\sin 6\gamma}-\frac{2}{3\sqrt{3}}, \qquad N_5 = -\frac{1}{2\cos 3\gamma}+\tfrac{1}{2}, \qquad (8)$$

And similarly, in $-S_2$, with γ replaced by $\frac{1}{2}\pi-\gamma$,

$$M_1 = \frac{1+\sin\gamma}{\cos\gamma}, \qquad M_2 = -\frac{\cos\gamma}{6\sin\gamma}+\frac{1+\cos 2\gamma}{-3\cos 3\gamma},$$

$$M_3+M_4 = \frac{2\sin 2\gamma}{\sqrt{3}\sin 6\gamma}-\frac{2}{3\sqrt{3}}, \qquad M_5 = -\frac{1}{-2\sin 3\gamma}+\tfrac{1}{2}, \qquad (9)$$

leading to the same value as before of

$$\frac{K}{a} = \frac{S_1-S_2}{2\pi}. \qquad (10)$$

Thus for the symmetrical lens, as before,

$$\beta = \tfrac{3}{4}\pi,\ \gamma = \tfrac{1}{4}\pi,\ N_1 = \frac{\sqrt{2}+1}{6},\ N_2 = -\tfrac{1}{6}+\frac{\sqrt{2}}{3},\ N_3+N_4 = -\frac{2}{\sqrt{3}}-\frac{2}{3\sqrt{3}},$$

$$N_5 = \frac{1}{\sqrt{2}}+\tfrac{1}{2},\ S_1 = -S_2;\ \frac{K}{a} = 1+2\sqrt{2}-\frac{16}{3\sqrt{3}} = 0{\cdot}7492.$$

But the indeterminate form arises in the flat hemispherical lens; and here $b = \frac{1}{2}$, so that a factor cancels, and L_1, N_1, are absent.

11. The elliptic integral arises in the calculation of the P.F. and S.F. of the lens with $f = \frac{2}{3}, \frac{3}{4}$, and conducting angle 90°, 120°.

With $f = \frac{2}{3}$, and $\operatorname{ch}\frac{1}{3}\zeta = s, \quad \operatorname{ch}\frac{1}{3}u = b, \quad \cos\frac{1}{3}v = c,$

$$\operatorname{ch}\zeta - \operatorname{ch} u = 4s^3 - 3s - 4b^3 + 3b = \mathrm{S},$$

$$\operatorname{ch} u - \cos v = -\mathrm{S}(c) = -\mathrm{S}(-c) = 4b^3 - 3b - 4c^3 + 3c = -\Sigma,$$

$$\mathrm{C} = \int_b^\infty \frac{\sqrt{-\Sigma}}{\sqrt{\mathrm{S}}} \cdot \frac{2s\,ds}{s^2 - c^2} = \int \frac{\sqrt{-\Sigma}}{s-c} \cdot \frac{ds}{\sqrt{\mathrm{S}}} + \int \frac{\sqrt{-\Sigma}}{s+c} \cdot \frac{ds}{\sqrt{\mathrm{S}}}, \tag{1}$$

two complete elliptic integrals of the third kind in a standard form of Weierstrass; and here

$$g_2 = 3, \quad g_3 = \operatorname{ch} u, \quad \Delta = -27\operatorname{sh}^2 u, \quad \mathrm{J} = -\operatorname{cosech}^2 u, \quad \mathrm{J} - 1 = -\coth^2 u. \tag{2}$$

So too with $f = \frac{3}{4}$, and $\operatorname{ch}\frac{1}{4}\zeta = s, \quad \cos\frac{1}{4}v = c,$

$$\cos(\tfrac{1}{4}v + \tfrac{2}{3}\pi) = c_1, \quad \cos(\tfrac{1}{4}v + \tfrac{4}{3}\pi) = c_2,$$

$$\mathrm{C} = \int \sqrt{-\Sigma}\left(\frac{1}{s-c} + \frac{1}{s-c_1} + \frac{1}{s-c_2}\right)\frac{ds}{\sqrt{\mathrm{S}}}. \tag{3}$$

The reduction of these integrals to a final result depending on tabulated functions is an interesting application of Elliptic Function theory.

The gravitation P.F. and S.F. of a homogeneous lens have been worked out in the 'American Journal of Mathematics,' vols. 33 and 39.